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CERTAIN INTEGRATION FORMULAE USEFUL IN NUMERICAL COMPUTATION.

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In developing a function of a variable by Taylor's, Maclaurin's, or the Euler-Maclaurin formulae, the evaluation of the successive derivatives is a fundamental and in most cases indispensable operation. In many instances the higher derivatives become exceedingly complex and their evaluation correspondingly difficult. In such cases frequently the labor involved in finding results of the required degree of accuracy by these formulae becomes prohibitively great, and other methods are resorted to, involving usually some of the formulae of the Calculus of Finite Differences. The formulae given below, some of which are probably new and others old except as to the remainder term, will, it is believed, prove useful in many such cases, for the reason that the use of the higher derivatives is not involved except in the remainder term, in which term a roughly approximate value usually suffices. In certain cases it will be found desirable to develop the given function to a limited number of terms by Taylor's or Maclaurin's formula, to differentiate this series the number of times required to get an expression for the derivative involved in the remainder term, and then from this expression to determine roughly the value of the remainder term. Another method of indicating the degree of accuracy assured in cases where the value of the remainder term is not easily determined is to develop the function by two of the formulae given below, and then to compare the results obtained. It is usually safe to assume that the difference in the results obtained may be substituted for the remainder term in either of the two formulae employed. It will be noted that the remainder term, as given below, is expressed in a form similar to that frequently employed in Taylor's formula, and that it is therefore always possible to compare the accuracy assured with the accuracy assured by the use of Taylor's formula.

From the manner of deriving these formulae it will appear that it is not practicable to give all the formulae of like form, obtainable from the general formula (c), but the list here given is, probably, sufficiently complete to satisfy all ordinary needs. Should others be desired they may, of course, be obtained from the general formula. In fact, were it not for the considerable amount of labor involved in deriving these formulae one at a time, all these formulae could be thus obtained and there would be no occasion for appending the following rather formidable list.

These formulae are all based on the so-called Euler-Maclaurin formula, a semi-convergent formula which has been recently studied by E. Borel, E. W. Barnes,* and other European mathematicians, but to which but scanty reference is made in our American mathematical literature. It would seem a greater use of this formula in our American text books would be advisable.

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* Borel, *Lecons sur les sèries divergentes*, Chap. III.
Borel, *Acta Mathematica*, Vol. XX, page 360.
Barnes, *Quarterly Journal of Mathematics*, Vol. XXXV, pages 175-188.
Barnes, *Proc. London Mathematical Society*, Series 2, Vol. 3, page 253.
Cantor, *Geschichte der Mathematik*, Vol. 3, pages 635, 661.

William Fleetfoot Sheppard, in his article on "Mensuration" in the 11th edition of the Encyclopedia Brittannica, volume XVIII, pages 142-3-4, paragraphs 68 to 79 inclusive, gives a very satisfactory reference to this class of formulae, to which the reader is referred. In paragraph 71 he gives a general method of constructing some of these formulae, but it should be carefully noted that the subscripts of C which he uses are determined by a system of notation somewhat different from that herein used. In paragraph 79 he gives another general form involving an expression for the remainder, but the use of the formulae in general numerical work is not emphasized, and the reader is likely to infer that their use is confined chiefly to the subject under discussion.

The Euler-Maclaurin formula may be written,

$$f(a+x)=f(a)+\frac{x}{2m}\{f'(a)+f'(a+x)+2[f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})$$

$$+f'(a+\frac{3x}{m})\ldots.+f'(a+\frac{m-1}{m}x)]\}-\frac{B_1 x^2}{m^2.2!}[f''(a+x)-f''(a)]$$

$$+\frac{B_2 x^4}{m^4.4!}[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)]\ldots.$$

$$+(-1)^n\frac{B_n x^{2n}}{m^{2n}.(2n)!}[f^{2n}(a+x)-f^{2n}(a)]+S\ldots.(\text{a}).$$

B_n being Bernoulli's nth number.

Expressed briefly and for clearness in what follows, this becomes:

$$f(a+x)=F(x,\,m)+\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+\frac{k_8}{m^8}\ldots.+\frac{k_{2n}}{m^{2n}}+S\ldots.(\text{b}),$$

in which k_2, k_4, k_6, ..., k_{2n}, are independent of m, and S is an expression for the remainder after the general term.

If $(r+1)$ values of m be taken, r of the quantities k, say k_{2i}, $k_{2(i+1)}$, $k_{2(i+2)}$, ..., $k_{2(i+r-1)}$, may be eliminated between the $(r+1)$ resulting equations, and the result expressed in determinant form as follows:

$$f(a+x)=\frac{\begin{vmatrix} M_1+S_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ M_2+S_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1)} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot & \cdot \\ M_{(r+1)}+S_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot & \cdot \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}} \ldots (c),$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate values of $f(a+x)$, obtained by (b) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by employing all the terms preceding the term $\frac{k_{2i}}{m^{2i}}$, and where S_1, S_2, ..., $S_{(r+1)}$, represent the corresponding remainder terms, *i. e.* the remainder after the term $\frac{k_{2(i+r-1)}}{m^{2(i+r-1)}}$ in each of the $(r+1)$ equations.

Splitting up the determinant of the numerator into two other similar determinants, one containing the M elements and the other the S elements, we have, by letting D represent the determinant,

$$f(a+x)=D_M/D+D_S/D \ldots (d).$$

To rid the indices of the elements of these determinants of the minus sign, divide each element in a row by the last element in that row. Now reduce the order by taking the co-factors of all the elements in the first column. These co-factors have evidently as factors $(m^2_u-m^2_v)$, u and v taking any value assumed by the subscripts of m, except that of the element, the co-factor of which is involved, and except $u=v$. Their evaluation follows readily.

The remainders, S, used in determining the remainders in the following formulae, were obtained from the solution of problem 237, Calculus, given in the June-July, 1907, number of THE AMERICAN MATHEMATICAL MONTHLY, to which reference must be had for an understanding of the method used in their derivation. Each remainder term in the following is dependent for its value on but *one* higher derivative, and for that reason the limits of its value are fixed, while the form given by Mr. Sheppard is a function of *all* derivatives higher than a certain derivative, and usually becomes *divergent* if a sufficiently large number of terms be taken.

Instead of retaining in the following formulae the M_1, M_2, ..., $M_{(r+1)}$, used in (c), it is convenient to employ a slightly different notation, viz., C_m represents the chordal area of the curve $y=f'(a+x)$, where the value of y is taken at each of the $(m+1)$ equidistant intervals, 0, x/m, $2x/m$, $3x/m$, ..., $(m-1)x/m$ and x, thus

$$C_m = \frac{x}{m}\left[\frac{f'(a)+f'(a+x)}{2} + f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \ldots + f'\left(a+\frac{(m-1)}{m}x\right)\right].$$

In the formulae involving y, the first extreme ordinate is invariably y_o, and the last extreme ordinate y_n, where n has the highest value of the subscript given in that formula. The distances of the intermediate ordinates y_r (where $o<r<n$) and the last ordinate y_n from the first extreme ordinate are *proportional to their subscripts.* The ordinates are therefore not *all* equidistant *in cases where the smaller values of m are not all factors of the largest value.*

Where C is given more than one subscript, thus $C_{m,t}$, it is understood that to the expression for the chordal area C_m has been added the sum of the first t terms of the series $\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+$ etc., in (b), where

$$k_{2n} = +(-1)^n \frac{B_n x^{2n}}{(2n)!}[f^{2n}(a+x) - f^{2n}(a)],$$

B_n being Bernoulli's nth number.

$$A \equiv \int_0^x f'(a+x)\,dx \equiv \int_0^x y\,dx, \text{ and } R_{2n} \equiv \frac{x^{2n}}{(2n)!}[f^{2n}(a+\theta_1 x) - f^{2n}(a+\theta_2 x)],$$

where both θ_1 and θ_2 have some value greater then zero and less than unity.

$A=\frac{1}{3}(4C_2-C_1)\pm\frac{4}{15}R_4$ . . . . (1).

$A=\frac{1}{8}(9C_3-C_1)\pm\frac{4}{45}R_4$ . . . . (2).

$A=\frac{1}{45}(64C_4-20C_2+C_1)\pm\frac{3}{70}R_6$ . . . . (3).

$A=\frac{1}{120}(243C_3-128C_2+5C_1)\pm\frac{2}{21}R_6$ . . . . . . (4).

$A=\frac{1}{2520}(8192C_4-6561C_3+896C_2-7C_1)\pm.0229R_8$ . . . . (5).

$A=\frac{1}{350}(512C_{12}-175C_6+14C_3-C_2)\pm.000,14R_8$ . . . . (6).

$A=\frac{1}{35}(56C_{12}-28C_6+8C_4-C_3)\pm.000,099R_8$ . . . . . (7).

$A=\frac{1}{10}(15C_6-6C_3+C_2)\pm.006,62R_6$ . . . . (8).

$A=\frac{1}{210}(567C_6-512C_4+162C_3-7C_2)\pm.002,55R_8$ . . . . (9).

$A=\frac{1}{4200}(10,368C_6-8,192C_4+2,025C_3-C_1)\pm.001,93R_8$ . . . (10).

$$A=\tfrac{1}{3150}(6{,}561C_6-3{,}584C_4+175C_2-2C_1)\pm .003{,}77R_8 \tag{11}$$

$$A=\tfrac{1}{840}(1{,}296C_6-567C_3+112C_2-C_1)\pm .008{,}39R_8 \tag{12}$$

$$A=\tfrac{1}{12600}(34{,}992C_6-32{,}768C_4+10{,}935C_3-560C_2+C_1)\pm .002{,}74R_{10} \tag{13}$$

$$A=\tfrac{1}{2835}(4{,}096C_8-1{,}344C_4+84C_2-C_1)\pm .002{,}02R_8 \tag{14}$$

$$A=\tfrac{1}{8731800}(33{,}554{,}432C_8-31{,}177{,}872C_6+7{,}569{,}408C_4-1{,}240{,}029C_3 \\ +25{,}872C_2-11C_1)\pm .000{,}225R_{12} \tag{15}$$

$$A=\tfrac{1}{1801800}(2{,}985{,}984C_{12}-1{,}667{,}952C_6+585{,}728C_4-104{,}247C_3 \\ +2{,}288C_2-C_1)\pm .000{,}095{,}6R_{12} \tag{16}$$

$$A=\tfrac{1}{1050}(1{,}728C_{12}-945C_6+320C_4-54C_3+C_2)\pm .000{,}071{,}2R_{10} \tag{17}$$

$$A=\tfrac{1}{280}(384C_6-105C_3+C_1)\pm .006{,}05R_6 \tag{18}$$

$$A=C_6+\tfrac{13}{35}(C_6-C_3)-\tfrac{1}{280}(C_3-C_1)\pm .006{,}05R_6 \tag{18a}$$

$$A=\tfrac{1}{210}(243C_6-35C_2+2C_1)\pm .017{,}1R_6 \tag{19}$$

$$A=C_6+\tfrac{11}{70}(C_6-C_2)-\tfrac{1}{105}(C_2-C_1)\pm .017{,}1R_6 \tag{19a}$$

$$A=\tfrac{1}{45}(64C_8-20C_4+C_2)\pm .001{,}34R_6 \tag{20}$$

$$A=\tfrac{1}{722925}(1{,}048{,}576C_{16}-348{,}160C_8+22{,}848C_4-340C_2+C_1) \\ \pm .000{,}030{,}2R_{10} \tag{21}$$

$$A=\tfrac{1}{65270205000}(110{,}075{,}314{,}176C_{24}-64{,}900{,}362{,}240C_{12}+25{,}081{,}937{,}920C_8 \\ -5{,}178{,}990{,}960C_6+202{,}076{,}160C_4-9{,}808{,}695C_3+38{,}640C_2-C_1) \\ \pm .000{,}000{,}134R_{16} \tag{22}$$

$$A=\tfrac{1}{2376}(3{,}200C_{10}-825C_5+C_1)\pm .000{,}624R_6 \tag{23}$$

$$A=\tfrac{1}{2010993768}(3{,}648{,}810{,}176C_{18}-1{,}674{,}098{,}987C_9+36{,}837{,}504C_6 \\ -575{,}586C_3+20{,}672C_2-11C_1)\pm .000{,}000{,}736R_{12} \tag{24}$$

Formulae involving *dy/dx*.

$$A=\tfrac{1}{15}(16C_{2,1}-C_{1,1})\pm .115R_6 \tag{25}$$

$$A=C_{2,1}+\tfrac{1}{15}(C_{2,1}-C_{1,1})\pm .115R_6 \tag{25a}$$

$$A=\tfrac{1}{80}(81C_{3,1}-C_{1,1})\pm .019{,}1R_6 \tag{26}$$

$$A=C_{3,1}+\tfrac{1}{80}(C_{3,1}-C_{1,1})\pm .019{,}1R_6 \tag{26a}$$

$$A=\tfrac{1}{945}(1{,}024C_{4,1}-80C_{2,1}+C_{1,1})\pm .005{,}9R_8 \tag{27}$$

$$A=C_{4,1}+\tfrac{79}{945}(C_{4,1}-C_{2,1})-\tfrac{1}{945}(C_{2,1}-C_{1,1})\pm .005{,}9R_8 \tag{27a}$$

$$A=\tfrac{1}{1680}(2{,}187C_{3,1}-512C_{2,1}+5C_{1,1})\pm .019{,}6R_8 \tag{28}$$

$$A=\tfrac{1}{75600}(131{,}072C_{4,1}-59{,}049C_{3,1}+3{,}584C_{2,1}-7C_{1,1})\pm .002{,}88R_{10} \tag{29}$$

$$A=\tfrac{1}{33775}(36{,}864C_{12,1}-3{,}150C_{6,1}+63C_{3,1}-2C_{2,1})\pm .000{,}002{,}76R_{10} \tag{30}$$

$A=\frac{1}{245}(270C_{6,1}-27C_{3,1}+2C_{2,1})\pm.000{,}389R_8$ . . . . . . (31).

$A=\frac{1}{6825}(10{,}206C_{6,1}-4{,}096C_{4,1}+729C_{3,1}-14C_{2,1})\pm.000{,}151R_{10}$ . . (32).

$A=\frac{1}{260400}(373{,}248C_{6,1}-131{,}072C_{4,1}+18{,}225C_{3,1}-C_{1,1})\pm.000{,}118R_{10}$. (33).

$A=\frac{1}{89775}(118{,}098C_{6,1}-28{,}672C_{4,1}+350C_{2,1}-C_{1,1})\pm.000{,}25R_{10}$ . . (34).

$A=\frac{1}{42000}(46{,}656C_{6,1}-5{,}103C_{3,1}+448C_{2,1}-C_{1,1})\pm.000{,}633R_{10}$ . . (35).

$A=\frac{1}{831600}(1{,}259{,}712C_{6,1}-524{,}288C_{4,1}+98{,}415C_{3,1}-2{,}240C_{2,1}+C_{1,1})$
$\pm.000{,}202R_{12}$ . . . . (36).

$A=\frac{1}{240975}(262{,}144C_{8,1}-21{,}504C_{4,1}+336C_{2,1}-C_{1,1})\pm.000{,}089{,}8R_{10}$. (37).

$A=\frac{1}{1135134000}(2{,}147{,}483{,}648C_{8,1}-1{,}122{,}403{,}392C_{6,1}+121{,}110{,}528C_{4,1}$
$-11{,}160{,}261C_{3,1}+103{,}488C_{2,1}-11C_{1,1})\pm.000{,}010{,}7R_{14}$ . . (38).

$A=\frac{1}{378378000}(429{,}981{,}696C_{12,1}-60{,}046{,}272C_{6,1}+9{,}371{,}648C_{4,1}$
$-938{,}223C_{3,1}+9{,}152C_{2,1}-C_{1,1})\pm.000{,}002{,}8R_{14}$ . . . (39).

$A=\frac{1}{109725}(124{,}416C_{12,1}-17{,}010C_{6,1}+2{,}560C_{4,1}-243C_{3,1}+2C_{4,1})$
$\pm.000{,}001{,}66R_{12}$ . . . . (40).

$A=\frac{1}{299376}(320{,}000C_{10,1}-20{,}625C_{5,1}+C_{1,1})\pm.000{,}014{,}3R_8$ . . . (41).

Formulae involving $\frac{dy}{dx}$, $\frac{d^3y}{dx^3}$ and $\frac{d^5y}{dx^5}$.

$A=\frac{1}{63}(64C_{2,2}-C_{1,2})\pm.078{,}4R_8$ . . . . (42).

$A=C_{2,2}+\frac{1}{63}(C_{2,2}-C_{1,2})\pm.078{,}4R_8$ . . . . (42a).

$A=\frac{1}{728}(729C_{3,2}-C_{1,2})\pm.006{,}03R_8$ . . . . (43).

$A=C_{3,2}+\frac{1}{728}(C_{3,2}-C_{1,2})\pm.006{,}03R_8$ . . . . (43a).

$A=\frac{1}{16065}(16384C_{4,2}-320C_{2,2}+C_{1,2})\pm.001{,}31R_{10}$ . . . . . (44).

$A=\frac{1}{17640}(19{,}683C_{3,2}-2{,}048C_{2,2}+5C_{1,2})\pm.007{,}05R_{10}$ . . . (45).

$A=C_{3,2}+\frac{227}{1960}(C_{3,2}-C_{2,2})-\frac{1}{3528}(C_{2,2}-C_{1,2})\pm.007{,}05R_{10}$ . . . (45a).

$A=\frac{1}{1580040}(2{,}097{,}152C_{4,2}-531{,}441C_{3,2}+14{,}336C_{2,2}-7C_{1,2})\pm.000{,}669R_{12}$ (46).

$A=\frac{1}{5195575}(5{,}308{,}416C_{12,2}-113{,}400C_{6,2}+567C_{3,2}-8C_{2,2})$
$\pm.000{,}000{,}086{,}3R_{12}$ . . . . (47).

$A=\frac{1}{9485}(9{,}720C_{6,2}-243C_{3,2}+8C_{2,2})\pm.000{,}037{,}9R_{10}$ . . . . . (48).

$A=C_{6,2}+\frac{47}{1897}(C_{6,2}-C_{3,2})-\frac{8}{9485}(C_{3,2}-C_{2,2})\pm.000{,}037{,}9R_{10}$ . . (48a).

$A=\frac{1}{378378000}(45{,}349{,}632C_{6,2}-8{,}388{,}608C_{4,2}+885{,}735C_{3,2}-8{,}960C_{2,2}$
$+C_{1,2})\pm.000{,}027{,}3R_{14}$ . . . . (49).

$A=\frac{1}{1635480}(1{,}679{,}616C_{6,2}-45{,}927C_{3,2}+1{,}792C_{2,2}-C_{1,2})\pm.000{,}079R_{12}$ (50).

$$A=\tfrac{1}{16434495}(16{,}777{,}216C_{8,2}-344{,}064C_{4,2}+1{,}344C_{2,2}-C_{1,2})$$
$$\pm.000{,}006{,}39R_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51).$$

$$A=\tfrac{1}{5989723740 0}(61{,}917{,}364{,}224C_{12,2}-2{,}161{,}665{,}792C_{6,2}+149{,}946{,}368C_{4.2}$$
$$-8{,}444{,}007C_{3,2}+36{,}608C_{2,2}-C_{1,2})\pm.000{,}000{,}108R_{16} \quad . \quad . \qquad (52).$$

$$A=\tfrac{1}{255}(256C_{2,3}-C_{1,3})\pm.073{,}1R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53).$$

$$A=C_{2,3}+\tfrac{1}{255}(C_{2,3}-C_{1,3})\pm.073{,}1R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53a).$$

$$A=\tfrac{1}{6560}(6{,}561C_{3,3}-C_{1,3})\pm.002{,}53R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54).$$

$$A=C_{3,3}+\tfrac{1}{6560}(C_{3,3}-C_{1,3})\pm.002{,}53R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54a).$$

$$A=\tfrac{1}{260865}(262{,}144C_{4,3}-1{,}280C_{2,3}+C_{1,3})\pm.000{,}391R_{12} \quad . \quad . \quad . \qquad (55).$$

$$A=C_{4,3}+\tfrac{1}{260865}[1{,}279(C_{4,3}-C_{2,3})-(C_{2,3}-C_{1,3})]\pm.000{,}391R_{12} \quad . \quad . \qquad (55a).$$

$$A=\tfrac{1}{60060000}(60{,}466{,}176C_{6,3}-413{,}343C_{3,3}+7{,}168C_{2,3}-C_{1,3})$$
$$\pm.000{,}013{,}3R_{14} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56).$$

Formulae expressed in terms of ordinates.

$$A=\frac{x}{6}[(y_0+y_2)+4y_1]\pm\tfrac{4}{15}R_4 \text{ (Simpson's first formula)} \quad . \quad . \qquad (1r).$$

$$A=\frac{x}{8}[(y_0+y_3)+3(y_1+y_2)]\pm\tfrac{4}{45}R_4 \text{ (Simpson's second formula)} \quad . \qquad (2r).$$

$$A=\frac{x}{90}[7(y_0+y_4)+32(y_1+y_3)+12y_2]\pm\tfrac{3}{70}R_6 \quad . \quad . \quad . \quad . \quad . \qquad (3r).$$

$$A=\frac{x}{120}[11(y_0+y_6)+81(y_2+y_4)-64y_3]\pm\tfrac{2}{21}R_6 \quad . \quad . \quad . \quad . \qquad (4r).^*$$

$$A=\frac{x}{2520}[151(y_0+y_{12})+2{,}048(y_3+y_9)+2{,}496y_6-2{,}187(y_4+y_8)$$
$$\pm.022{,}9R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5r).^*$$

$$A=\frac{x}{2100}[53(y_0+y_{12})+256(y_1+y_3+y_5+y_7+y_9+y_{11})+81(y_2+y_{10})$$
$$+78y_6+109(y_4+y_8)]\pm.000{,}14R_8 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6r).$$

$$A=\frac{x}{210}[5(y_0+y_{12})+28(y_1+y_5+y_7+y_{11})+40(y_3+y_9)+12y_6$$
$$-2(y_4+y_8)]\pm.000{,}099R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7r).$$

$$A=\frac{x}{20}[(y_0+y_2+y_4+y_6)+5(y_1+y_5)+6y_3]\pm.006{,}62R_6 \text{ (Weddle's}$$
$$\text{formula)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8r).$$

$$A=\frac{x}{420}[17(y_0+y_{12})+189(y_2+y_{10})+297(y_4+y_8)-256(y_3+y_9)-74y_6]$$
$$\pm.002{,}55R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9r).^*$$

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*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$A=\frac{x}{5670}[217(y_0+y_8)+1{,}024(y_1+y_3+y_5+y_7)+352(y_2+y_6)+436y_4]$$
$$\pm .002{,}02R_8 \quad \ldots \quad (14r).$$

$$A=\frac{x}{840}[41(y_0+y_6)+216(y_1+y_5)+27(y_2+y_4)+272y_3]\pm .008{,}39R_8 \quad (12r).$$

$$A=\frac{x}{280}[15(y_0+y_6)+64(y_1+y_3+y_5)+29(y_2+y_4)]\pm .006{,}05R_6 \quad . \quad (18r).$$

$$A=\frac{x}{420}[25(y_0+y_6)+81(y_1+y_2+y_4+y_5)+46y_3]\pm .017{,}1R_6 \quad . \quad . \quad (19r).$$

$$A=\frac{x}{2100}[49(y_0+y_{12})+288(y_1+y_5+y_7+y_{11})+448(y_3+y_9)-27(y_2+y_{10})$$
$$-63(y_4+y_8)+134y_6]\pm .000{,}071{,}2R_{10} \quad \ldots \quad (17r).$$

$$A=\frac{x}{1801800}[41{,}833(y_0+y_{12})+248{,}832(y_1+y_5+y_7+y_{11})+395{,}264(y_3+y_9)$$
$$+118{,}416y_6-29{,}160(y_2+y_{10})-63{,}909(y_4+y_8)]\pm .000{,}095{,}6R_{12} \quad (16r).$$

Formulae expressed in terms of ordinates and the derivatives of the extreme ordinates, wherein $E_n\equiv[f^n(a+x)-f^n(a)]$.

$$A=\frac{x}{30}[7(y_0+y_2)+16y_1]-\frac{x^2}{60}E_2\pm .115R_6 \quad \ldots \quad (25s).$$

$$A=\frac{x}{80}[13(y_0+y_3)+27(y_1+y_2)]-\frac{x^2}{120}E_2\pm .019{,}1R_6 \quad \ldots \quad (26s).$$

$$A=\frac{x}{1890}[217(y_0+y_4)+512(y_1+y_3)+432y_2]-\frac{x^2}{252}E_2\pm .005{,}9R_8 \quad . \quad (27s).$$

$$A=\frac{x}{1680}[239(y_0+y_6)+729(y_2+y_4)-256y_3]-\frac{x^2}{168}E_2\pm .019{,}6R_8 \quad . \quad (28s).^*$$

$$A=\frac{x}{75600}[7{,}435(y_0+y_{12})+32{,}768(y_3+y_9)+34{,}560y_6-19{,}683(y_4+y_8)]$$
$$-\frac{x^2}{360}E_2\pm .002{,}88R_{10} \quad \ldots \quad (29s).^*$$

$$A=\frac{x}{13650}[913(y_0+y_{12})+3{,}402(y_2+y_{10})-2{,}048(y_3+y_9)+3{,}888(y_4+y_8)$$
$$+1{,}340y_6]-\frac{x^2}{780}E_2\pm .000{,}151R_{10} \quad \ldots \quad (32s).^*$$

$$A=\frac{x}{42000}[3{,}149(y_0+y_6)+7{,}776(y_1+y_5)+6{,}075(y_2+y_4)+8{,}000y_3]$$
$$-\frac{x^2}{600}E_2\pm .000{,}633R_{10} \quad \ldots \quad (35s).$$

$$A=\frac{x}{378378000}[13{,}929{,}421(y_0+y_{12})+35{,}831{,}808(y_1+y_5+y_7+y_{11})$$
$$+25{,}824{,}096(y_2+y_{10})+38{,}174{,}720(y_3+y_9)+25{,}511{,}355(y_4+y_8)$$

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*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$+28,171,584y_6] - \frac{x^2}{2520}E_2 \pm .000,002,8R_{14} \quad . \quad . \quad . \quad . \quad (39s).$$

$$A = \frac{x}{126}[31(y_0+y_2)+64y_1] - \frac{5x^2}{252}E_2 + \frac{x^4}{15120}E_4 \pm .078,4R_8 \quad . \quad . \quad (42t).$$

$$A = \frac{x}{728}[121(y_0+y_3)+243(y_1+y_2)] - \frac{5x^2}{546}E_2 + \frac{x^4}{65520}E_4 \pm .006,03R_8 \quad (43t).$$

$$A = \frac{x}{32130}[3,937(y_0+y_4)+8,192(y_1+y_3)+7,872y_2] - \frac{x^2}{204}E_2 + \frac{x^4}{257040}E_4$$
$$\pm .001,31R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44t).$$

$$A = \frac{x}{1635480}[132,761(y_0+y_6)+279,936(y_1+y_5)+264,627(y_2+y_4)$$
$$+280,832y_3] - \frac{25x^2}{11682}E_2 + \frac{x^4}{1401840}E_4 \pm .000,079R_{12} \quad . \quad . \quad (50t).$$

The well known *Simpson's rule* and *Weddle's rule* involve the successive use of *Simpson's formula* and *Weddle's formula* respectively. (See formulae (1*r*), (2*r*) and (8*r*) above). Similar use may, of course, be made of other of the foregoing formulae in *y*.

Still other formulae may be had in these ordinates by eliminating between two or more equations involving the same set, as between (12*r*) and (18*r*). The new formulae thus formed are usually less accurate than those above given, having been formed by considering the remainders zero or negligible, but may be advantageously used in cases where the values of certain ordinates are not readily obtainable.

For ready reference the numerical coefficients in some of the foregoing formulae and their logarithms are given below in decimal form. There may be an error of ±1 in the last decimal place, as results were obtained by using a 10-place table of logarithms.

Formula (15)

C_8	3.842,785,222	.584,646,111,9
C_6	3.570,612,245	.552,742,690,0
C_4	.866,878,308	9.937,958,135,2—10
C_3	.142,012,987	9.152,328,062,0—10
C_2	.002,962,963	7.471,726,222,9—10
C_1	.000,001,260	4.100,288,905,5—10

Formula (21)

C_{16}	1.450,463,049	.161,506,669,3
C_8	.481,599,060	9.682,685,630,1—10
C_4	.031,604,938	8.499,754,946,5—10
C_2	.000,470,312	6.672,385,673,4—10
C_1	.000,001,383	4.140,906,756,4—10

Formula (22)		
C_{24}	1.686,455,775	.226,974,956,5
C_{12}	.994,333,666	9.997,532,144,1−10
C_8	.384,278,522	9.584,646,111,9—10
C_6	.079,346,939	8.899,530,176,2—10
C_4	.003,095,994	7.490,800,103,7−10
C_3	.000,150,278	6.176,896,253,7—10
C_2	.000,000,592	3.772,322,141,0−10
C_1	.000,000,000	9.185,285,023,3−20
Formula (24)		
C_{18}	1.814,431,370	.258,740,545,7
C_9	.832,473,483	9.920,370,408,9—10
C_6	.018,318,060	8.262,879,471,3−10
C_3	.000,286,220	6.456,699,497,3−10
C_2	.000,010,279	5.011,971,771,6—10
C_1	.000,000,005	1.737,981,960,5—10
Formula (39)		
$C_{12,1}$	1.136,381,334	.055,524,091,1
$C_{6,1}$	.158,693,878	9.200,560,171,9−10
$C_{4,1}$	.024,767,952	8.393,890,090,8−10
$C_{3,1}$	.002,479,592	7.394,380,197,9—10
$C_{2,1}$	.000,024,187	5.383,590,134,1−10
$C_{1,1}$	.000,000,003	1.422,074,122,7−10
Formula (50)		
$C_{6,2}$	1.026,986,573	.011,564,765,5
$C_{3,2}$	.028,081,664	8.448,422,840,3—10
$C_{2,2}$	.001,095,703	7.039,692,767,8−10
$C_{1,2}$	.000,000,611	3.786,354,762,5—10
Formula (51)		
$C_{8,2}$	1.020,853,761	.008,963,532,4
$C_{4,2}$	.020,935,477	8.320,882,870,6—10
$C_{2,2}$	.000,081,779	5.912,642,905,3−10
$C_{1,2}$	.000,000,061	2.784,243,636,6—10
Formula (52)		
$C_{12,2}$	1.033,726,545	.014,405,668,3
$C_{6,2}$	.036,089,574	8.557,381,757,7—10
$C_{4,2}$	.002,503,394	7.398,529,158,3−10
$C_{3,2}$	.000,140,975	6.149,141,792,4—10
$C_{2,2}$	.000,000,611	3.786,169,210,6—10
$C_{1,2}$	.000,000,000	9.222,593,207,8−20

EXAMPLES.

Ex. 1. Construct a five-place table of the values of $\int_0^{n\pi}\frac{\sin x}{x}$ for all integral values of n from 1 to 10, inclusive.

Solution. $\int_0^{10\pi}\frac{\sin x}{x}dx=\sum_{n=0}^{n=10}\int_0^{\pi}\frac{\sin(n\pi+x)}{n\pi+x}dx.$ Similarly for $n=9$, 8, 7, etc.

Each value to be found may therefore be derived by the aid of the sum of all the preceding values, already found.

To determine which of the given formulae is sufficiently accurate and at the same time involves the minimum amount of labor is now advisable. Formula (17) has $\pm.0000712R_{10}$ for its remainder term, and, as seems probable, ought to be within the bounds of accuracy required. Expressed more fully this remainder term is

$$\pm\frac{712}{10^7}\cdot\frac{\pi^{10}}{10!}[f^{10}(x_1)-f^{10}(x_2)], \text{ where } 0\leqq x_1\leqq\pi \text{ and } 0\leqq x_2\leqq\pi.$$

Developing $\sin x/x$ into a power series in x, and differentiating nine times to obtain the tenth derivative, we get another power series in x, the value of which lies somewhere between 0 and $-\frac{1}{5}$ for $0\leqq x\leqq\pi$. Substituting $-\frac{1}{5}$ for the bracket expression above, we find that the remainder term has a value within the limits of $\pm.000{,}000{,}4$. (17) is therefore sufficiently accurate for this value of x, and it can be readily shown that its value for other values of x involved in the problem is less than its value between the limits here considered.

The chordal areas (represented generally by C with a subscript equal to the number of equidistant ordinates employed, less one, the first ordinate being taken at the point $x=0$ and the last at the point $x=X$, where X is the maximum value of x), involved in (17) are,

$$C_{12}=1.850{,}119,\quad C_6=1.844{,}651,\quad C_4=1.835{,}508,$$
$$C_3=1.822{,}636, \text{ and } C_2=1.785{,}398.$$

Substituting in (17), $\int_0^{\pi}\frac{\sin x}{x}dx=1.851{,}937.$

To check the accuracy of the numerical work done, these values of C may be substituted in (8), and, by doing so, it is found that the same result is obtained by either formula. It is therefore probably safe to employ the simpler formula in computing the remaining terms sought.

It is also instructive to apply some of the other formulae involving the same values of *C*, *e. g.*, (9), (10), (11), (13), (16), etc., as well as those involving the second, fourth and sixth derivatives of the function sought, such as (43*a*), (45*a*), (48*a*), (55*a*), etc.

Ex. 2. Find the common logarithm of 2 from the identity,

$$10\log 2-\log 1000=M\int_0^{.024}\frac{dx}{1+x},$$

M being the modulus of the common system of logarithms.

Here the remainder term is easily evaluated, and the selection of the formula to be used is determined largely by the degree of accuracy required. Employing (35*s*) we get, log2=.301,029,995,663,981,195,213,738,269,3, which we know from the form of the remainder must be correct to not less than 21 decimal places. Expressed in terms of the formula after some simple reductions have been made, the development is,

$$\log 2=\tfrac{3}{10}+\frac{M}{10}\left(\frac{253}{3{,}500}\left[\frac{3{,}149}{64{,}000}+\frac{7{,}776}{64{,}005}+\frac{6{,}075}{64{,}008}\right]+\frac{8}{1{,}771}-\frac{2{,}277}{10^8.512}\right)$$
$$\pm.000{,}633R_{10}.$$

Using (39s) a result correct to thirty places of decimals would have been assured. Should still greater accuracy be desired, the step-by-step process of Ex. 1 may be used.

Ex. 3. Similarly, the logarithm of 2 known, the common logarithm of 3, correct to forty-seven decimal places, may be obtained by (39s) from the identity,

$$8\log 3+\log 10-16\log 2=M\int_0^{\frac{37}{32768}}\frac{dx}{1+x}.$$

Ex. 4. Obtain the value of $\int_0^{\frac{1}{2}\pi}\frac{x\,dx}{\sin x(1+\frac{4}{25}\cos^2 x)^{\frac{3}{2}}}$ correct to six decimal places.